TWO P-ADIC MEROMORPHIC FUNCTIONS SHARING A FEW SMALL FUNCTIONS I.M.

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This paper is dedicated to the memory of Professor Walter Hayman

Abstract. A new Nevanlinna theorem on q p-adic small functions is given. Let $f,\ g$, be two meromorphic functions on a complete ultrametric algebraically closed field $\mathbb{K}$ of characteristic 0, or two meromorphic functions in an open disk of $\mathbb{K}$, that are not quotients of bounded analytic functions by polynomials. If f and g share 7 small meromorphic functions I.M., then $f = g$.

Better results hold when f and g satisfy some property of growth. Particularly, if f and g have finitely many poles or finitely many zeros and share 3 small meromorphic functions I.M., then $f = g$.

1. Generalities

Definitions: We denote by $\mathbb{K}$ a complete ultrametric algebraically closed field of characteristic 0. Given $a \in \mathbb{K}$ and $R \in \mathbb{R}_+$, we denote by $d(0, R^-)$ the disk $\{x \in \mathbb{K} \mid |x| < R\}$, by D the set $d(0, R^-)$ and by E the set $\mathbb{K} \setminus d(0, R^-) = \{x \in \mathbb{K} \mid |x| \geq R\}$.

We denote by $\mathcal{A}(\mathbb{K})$ (resp. $\mathcal{A}(D)$, the algebra of analytic functions in $\mathbb{K}$ (resp. in D, i.e. the $\mathbb{K}$-algebra of power series converging in D). Next, we denote by $\mathcal{A}(E)$ the $\mathbb{K}$-algebra of Laurent series converging in E [3], [4], [9].

Next, we denote by $\mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(D)$) the field of fractions of $\mathcal{A}(\mathbb{K})$ (resp. $\mathcal{A}(D)$). We will also denote by $\mathcal{A}_u(D)$ the $\mathbb{K}$-algebra of unbounded analytic functions in D and by $\mathcal{M}_u(D)$ the set of meromorphic functions in D that are not a quotient of two bounded analytic functions in D. Finally, we denote by $\mathcal{M}(E)$ the field of fractions of $\mathcal{A}(E)$.

We have to introduce the counting function of zeros and poles of a meromororphic function f, counting or not multiplicity. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

Definitions: Let $f \in \mathcal{M}(d(0, r)$ and for every $a \in d(0, r)$, let $\omega(f)$ be the multiplicity order of 0 if 0 is a zero of f, and $\omega(f) = 0$, else. Next, let $\theta(f)$ be the multiplicity order of 0 if 0 is a pole of f, and $\theta(f) = 0$ else.

We denote by $Z(r, f)$ *the counting function of zeros of f in $d(0, r)$* in the following way.

Let $(a_n),\ 1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, of respective order s_n.

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We set $Z(r,f) = \max(\omega(f)\log r + \sum_{n=1}^{\sigma(r)} s_n(\log r - \log|a_n|)$ and so, $Z(r,f)$ is called *the counting function of zeros of f in $d(0,r)$, counting multiplicity.*

In order to define the counting function of zeros of f ignoring multiplicity, we put $\overline{\omega}(f) = 1$ if $\omega(f) > 0$ and $\overline{\omega}(f) = 0$ else Now, we denote by $\overline{Z}(r,f)$ *the counting function of zeros of f ignoring multiplicity*:

$\overline{Z}(r,f) = \overline{\omega_0}(f)\log r + \sum_{n=1}^{\sigma(r)}(\log r - \log|a_n|)$ and so, $\overline{Z}(r,f)$ is called *the counting function of zeros of f in $d(0,r)$ ignoring multiplicity.*

In the same way, considering the finite sequence (b_n), $1 \leq n \leq \tau(r)$ of poles of f such that $0 < |b_n| \leq r$, with respective multiplicity order t_n, we put

$N(r,f) = \theta(f)\log r + \sum_{n=1}^{\tau(r)} t_n(\log r - \log|b_n|)$ and then $N(r,f)$ is called *the counting function of the poles of f, counting multiplicity.*

Next, in order to define the counting function of poles of f ignoring multiplicity, we set

$\overline{N}(r,f) = \overline{\theta}(f)\log r + \sum_{n=1}^{\tau(r)}(\log r - \log|b_n|)$ and then $\overline{N}(r,f)$ is called *the counting function of the poles of f, ignoring multiplicity.*

Now, we can define the characteristic function of f as $T(r,f) = \max(Z(r,f), N(r,f))$. Thus this definition applies to functions $f \in \mathcal{M}(d(0,R^-)$ as well as functions $f \in \mathcal{M}(\mathbb{K})$.

Consider now a function $f \in \mathcal{A}(E)$. By definition, the restriction of f to any annulus $R \leq |x| \leq S$ is an annalytic element in that annulus and hence has finitely many zeros in that annulus [3], [4], [9]. Similarly, a meromorphic function $f \in \mathcal{M}(E)$ has finitely many zeros and finitely many poles in the annulus $R \leq |x| \leq S$. That is summarized in Proposition 1.1:

Proposition 1.1 [1], [3], [4], [10] : *Let $f \in \mathcal{M}(E)$. If f has infinitely many zeros in E (resp. infinitely many poles in E), the set of zeros (resp. the set of poles) is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}|\alpha_n| = +\infty$. If f has no zero in E, then it is of the form $\sum_{-\infty}^{q} a_n x^n$ with $|a_q|r^q > |a_n|r^n \ \forall n < q, \ ,\forall r \geq R$.*

Proposition 1.2 [1], [3], [4], [10] : *Let $f \in \mathcal{M}(E)$ have no zero and no pole in E. There exists a unique integer $q \in \mathbb{Z}$ such that $x^{-q}f(x)$ has a limit $b \in \mathbb{K}^*$.*

Definitions: Let $f \in \mathcal{M}(E)$ have no zero and no pole in E. The integer $q \in \mathbb{Z}$ such that $x^{-q}f(x)$ has a limit $b \in \mathbb{K}^*$ is called *the Motzkin index of f* and f is called *a Motzkin factor* if $\lim_{|x|\to+\infty} x^{-q}f(x) = 1$ [1], [10].

Proposition 1.3 [1], [3], [4], [10] : *Let $f \in \mathcal{M}(E)$. Then f factorizes in a unique way in the form $f^S f^0$ where f^S is a Motzkin factor and $f^0 \in \mathcal{M}(E)$ has continuation to an element of $H(D)$ that has no zero in D.*

Notations: We will denote by $\mathcal{A}^c(E)$ the set of $f \in \mathcal{A}(E)$ having infinitely many zeros in E. Similarly, we will denote by $\mathcal{M}^c(E)$ the set of functions $f \in \mathcal{M}(E)$ which have infinitely many zeros or poles in E.

Thus we can define counting functions for zeros and poles in that way: Let $f \in \mathcal{M}(E)$ and, for every $r > R$, let $a_1, ..., a_{\sigma(r)}$ be the sequence of zeros of f in the annulus $R \leq |x| \leq r$, with $|a_j| \leq |a_{j+1}|$, $1 \leq j \leq \sigma(r)$, and let s_j be the order of a_j. Then we put $Z_R(r, f) = \sum_{j=1}^{\sigma(r)} s_j(\log(r) - \log(|a_j|))$ and $Z_R(r, f)$ is called *the counting function of zeros for f in $\mathcal{M}(E)$, counting multiplicity*. And we define $\overline{Z}_R(r, f) = \sum_{j=1}^{\sigma(r)}(\log(r) - \log(|a_j|))$ which is called *the counting function of zeros for f in $\mathcal{M}(E)$, ignoring multiplicity*.

Similarly, let $b_1, ..., b_{\tau(r)}$ be the sequence of zeros of f in the annulus $R \leq |x| \leq r$, with $|b_j| \leq |b_{j+1}|$, $1 \leq j \leq \tau(r)$ and let t_j be the order of b_j. Then we put $N_R(r, f) = \sum_{j=1}^{\tau(r)} t_j(\log(r) - \log(|b_j|))$ which is called *the counting function of poles for f in $\mathcal{M}(E)$, counting multiplicity* and we put $\overline{N}_R(r, f) = \sum_{j=1}^{\tau(r)}(\log(r) - \log(|b_j|))$ which is called *the counting function of poles for f in $\mathcal{M}(E)$, ignoring multiplicity*.

Now, we put $T_R(r, f) = \max(Z_R(r, f), N_R(r, f))$ and the function $T_R(r, f)$ is called *the characteristic function of $f \in \mathcal{M}(E)$*.

Remark: If we change the origin, the functions Z, N, T are not changed, up to an additive constant.

2. Nenalinna Theory

We have to recall the two main Theorems, applied to each domain of definition of meromorphic functions.

Theorem 2.1 (First Main Fundamental Theorem in a disk and in $\mathbb{K}$): *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. let $f,\ g \in \mathcal{M}(D)$). Then $T(r, f + b) = T(r, f) + O(1)$, $T(r, f+g) \leq T(r, f)+T(r, g)+O(1)$. Let $P(X) \in \mathbb{K}[X]$. Then $T(r, P(f)) = \deg(P)T(r, f) + O(1)$.*

Suppose now $f, g \in \mathcal{A}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{A}(D)$). Then $T(r, f) = Z(r, f)$, $Z(r, fg) = Z(r, f) + Z(r, g)$ and $T(r, f + g) \leq \max(T(r, f), T(r, g))$. Moreover, if $\lim_{r\to+\infty} T(r, f) - T(r, g) = +\infty$ then $T(r, f + g) = T(r, f)$ when r is big enough.

Theorem 2.2 (First Main Fundamental Theorem out of a hole): *Let $f,\ g \in \mathcal{M}(E)$. Then for every $b \in \mathbb{K}$, we have $T_R(r, f + b) = T_R(r, f) + O(\log(r))$, $(r \geq R)$ $T_R(r, f.g) \leq T_R(r, f)+T_R(r, g)+O(\log(r))$, $(r \geq R)$ $T_R(r, \frac{1}{f}) = T_R(r, f))$, $T_R(r, f+g) \leq T_R(r, f)+T_R(r, g)+O(\log(r))$ $(r \geq R)$ and $T_R(r, f^n) = nT_R(r, f)$.*

Moreover, if both f and g belong to $\mathcal{A}(E)$, then

$$T_R(r, f+g) \le \max(T_R(r,f), T_R(r,g)) + O(\log(r)) \ (r \ge R)$$

and $T_R(r, fg) = T_R(r,f) + T_R(r,g)$, $(r \ge R)$. Particularly, if $f \in \mathcal{A}(E)$, then $T_R(r, f+b) = T_R(r,f) + O(1)$ $(r \ge R)$. Given a polynomial $P(X) \in \mathbb{K}[X]$, then $T_R(r, P \circ f) = qT_R(r,f) + O(\log(r))$.

The Nevanlinna Theory is well known in $\mathbb{C}$ [11]. It was constructed in a field like $\mathbb{K}$ in the eighties and next, in a disk and out of a hole [2], [8], [4], [5].

Theorem 2.3 (Second Main Theorem in $\mathbb{K}$ and in a disk) [2], [4], [8]: *Let $\alpha_1, ..., \alpha_q \in \mathbb{K}$, with $q \ge 2$, let $S = \{\alpha_1, ..., \alpha_q\}$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then*

$$\sum_{j=1}^{q} \big(Z(r, f-\alpha_j) - \overline{Z}(r, f-\alpha_j)\big) \le T(r,f) + \overline{N}(r,f) - Z_0^S(r, f') - \log r + O(1) \ \forall r > 0$$

(resp. $\forall r < R$).

Theorem 2.4 (Second Main Theorem out of a hole) [5] : *Let $f \in \mathcal{M}(\mathbb{K})$ and let $a_1, ..., a_q \in \mathbb{K}$ be distinct. Then*
$(q-1)T_R(r,f) \le \sum_{j=1}^{q} Z_R(r, f-a_j) + O(\log(r))$ $\forall r \ge R$.

3. Small functions

Recall that given three functions ϕ, ψ, ζ defined in an interval $J =]R, +\infty[$ (resp. $J =]a, R[$), with values in $[0, +\infty[$, we shall write $\phi(r) \le \psi(r) + O(\zeta(r))$ if there exists a constant $b \in \mathbb{R}$ such that $\phi(r) \le \psi(r) + b\zeta(r)$. We shall write $\phi(r) = \psi(r) + O(\zeta(r))$ if $|\psi(r) - \phi(r)|$ is bounded by a function of the form $b\zeta(r)$.

Similarly, we shall write $\phi(r) \le \psi(r) + o(\zeta(r))$ if there exists a function h from $J =]R, +\infty[$ (resp. from $J =]a, R[$) to $\mathbb{R}$ such that $\lim_{r\to+\infty} \frac{h(r)}{\zeta(r)} = 0$ (resp. $\lim_{r\to R} \frac{h(r)}{\zeta(r)} = 0$) and such that $\phi(r) \le \psi(r) + h(r)$. And we shall write $\phi(r) = \psi(r) + o(\zeta(r))$ if there exists a function h from $J =]R, +\infty[$ (resp. from $J =]a, R[$) to $\mathbb{R}$ such that $\lim_{r\to+\infty} \frac{h(r)}{\zeta(r)} = 0$ (resp. $\lim_{r\to R} \frac{h(r)}{\zeta(r)} = 0$) and such that $\phi(r) = \psi(r) + h(r)$.

Definitions and notations: For each $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(D)$, resp. $f \in \mathcal{M}(E)$) we denote by $\mathcal{M}_f(\mathbb{K})$, (resp. $\mathcal{M}_f(D)$, resp. $\mathcal{M}_f(E)$) the set of functions $h \in \mathcal{M}(\mathbb{K})$, (resp. $h \in \mathcal{M}(D)$, resp. $\mathcal{M}(E)$) such that $T(r,h) = o(T(r,f))$ when r tends to $+\infty$ (resp. when r tends to R, resp. when r tends to $+\infty$). Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(D)$, $f \in \mathcal{A}(E)$) we shall denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(D)$, resp. $\mathcal{A}_f(E)$) the set $\mathcal{M}_f(\mathbb{K}) \cap \mathcal{A}(\mathbb{K})$, (resp. $\mathcal{M}_f(D) \cap \mathcal{A}(D)$, resp. $\mathcal{M}_f(E) \cap \mathcal{A}(E)$).

The elements of $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(D)$, resp. $\mathcal{M}_f(E)$) are called *small meromorphic functions with respect to f*, or *small functions* in brief. Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(D)$, resp. $f \in \mathcal{A}(E)$) the elements of $\mathcal{A}_f(\mathbb{K})$ (resp.

$\mathcal{A}_f(D)$, resp. $\mathcal{A}_f(E)$) are called *small analytic functions with respect to* f *or small functions in brief.*

Now we have several immediate results:

Theorem 3.1: *Let $a \in \mathbb{K}$ and $r > 0$. $\mathcal{A}_f(\mathbb{K})$ is a $\mathbb{K}$-subalgebra of $\mathcal{A}(\mathbb{K})$, $\mathcal{A}_f(D)$ is a $\mathbb{K}$-subalgebra of $\mathcal{A}(D)$, $\mathcal{A}_f(E)$ is a $\mathbb{K}$-subalgebra of $\mathcal{A}(E)$, $\mathcal{M}_f(\mathbb{K})$ is a subfield field of $\mathcal{M}(\mathbb{K})$, $\mathcal{M}_f(D)$ is a subfield of field of $\mathcal{M}(D)$ and $\mathcal{M}_f(E)$ is a subfield field of $\mathcal{M}(E)$. Moreover, $\mathcal{A}_b(D)$ is a sub-algebra of $\mathcal{A}_f(D)$ and $\mathcal{M}_b(D)$ is a subfield of $\mathcal{M}_f(D)$.*

Theorem 3.2: *Let $f \in \mathcal{M}(\mathbb{K})$, (resp.$f \in \mathcal{M}(D)$, resp. $f \in \mathcal{M}(E)$) and let $g \in \mathcal{M}_f(\mathbb{K})$, (resp. $g \in \mathcal{M}_f(D)$, resp. $g \in \mathcal{M}_f(E)$). Then $T(r, fg) = T(r, f) + o(T(r, f))$ and $T(r, \frac{f}{g}) = T(r, f) + o(T(r, f))$, (resp. $T(r, fg) = T(r, f) + o(T(r, f))$ and $T(r, \frac{f}{g}) = T(r, f) + o(T(r, f))$, resp. $T_R(r, fg) = T_R(r, f) + o(T_R(r, f))$ and $T_R(r, \frac{f}{g}) = T_R(r, f) + o(T_R(r, f))$).*

Here we can mention some precisions to Theorem 3.1 that will be useful later:

Theorem 3.3: *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. let $f \in \mathcal{A}_u(D)$, resp. let $f \in \mathcal{A}(E)$). Let $g,\ h \in \mathcal{A}_f(\mathbb{K})$ (resp. let $g,\ h \in \mathcal{A}_f(D)$, resp. let $g,\ h \in \mathcal{A}_f(E)$) with g and h not identically zero. If gh belongs to $\mathcal{A}_f(\mathbb{K})$ (resp. to $\mathcal{A}_f(D)$, resp. to $\mathcal{A}_f(E)$), then so do g and h.*

Theorem 3.4 : *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ (resp. let $f,\ g \in \mathcal{A}_u(D)$, resp. let $f,\ g \in \mathcal{A}(E)$) and let $q \in \mathbb{N}^*$. If $\frac{f}{g}$ is not a q-th root of 1, then $f^q - g^q$ does not belong to $\mathcal{A}_f(\mathbb{K})$ (resp. to $\mathcal{A}_f(D)$, resp. to $\mathcal{A}_f(E)$).*

Theorem 3.5 is a wide generalization of Theorem 2.1. It consists of the following claim: given a meromorphic function f and a rational function G of degree n whose coefficients are small functions with respect to f, then $T(r, G(f))$ is equivalent to $nT(r, f)$. The big difficulty consists of showing that $T(r, G(f))$ is not smaller than $nT(r, f)$. The proof, based on an elementary property of Bezout's Theorem, was given in $\mathbb{C}$ by F. Gackstatter and I. Laine [7] and was made in a field such as $\mathbb{K}$ by C.C. Yang and Peichu Hu [8].

Theorem 3.5: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(D)$, let $f \in \mathcal{M}(E)$). Let $G(Y) \in \mathcal{M}_f(\mathbb{K})(Y)$, (resp. $G \in \mathcal{M}_f(d(D))(Y)$, resp. $G(Y) \in \mathcal{M}_f(E)(Y)$) and let $n = \deg(G)$. Then $T(r, G(f)) = nT(r, f) + o(T(r, f))$, (resp. $T(r, G(f)) = nT(r, f) + o(T(r, f))$, resp. $T_R(r, G(f)) = nT_R(r, f) + o(T_R(r, f))$.*

Theorem 3.6: *Let $a \in \mathbb{K}$ and $r > 0$. Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. $f \in \mathcal{M}_u(D)$, resp. $f \in \mathcal{M}^c(E)$). Then, f is transcendental over $\mathcal{M}_f(\mathbb{K})$ (resp. over $\mathcal{M}_f(D)$, resp. over $\mathcal{M}_f(E)$).*

Proof. Suppose there exists a polynomial $P(Y) = \sum_{j=0}^{n} a_j Y^j \in \mathcal{M}_f(\mathbb{K})[Y] \neq 0$ such that $P(f) = 0$. If f belongs to $\mathcal{M}_u(d(a, R^-))$ we may obviously suppose that $a = 0$. By Theorem 3.6 we have $T(r, a_n f^n) = nT(r, f) + o(T(r, f))$ whenever f belongs to $\mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ or to $\mathcal{M}_f(d(0, R^-))$ and then $T_R(r, a_n f^n) = nT_R(r, f) + o(T_R(r, f))$ whenever f belongs to $\mathcal{M}^c(\mathbb{K}$, whereas $T(r, \sum_{j=0}^{n-1} a_j f^j) = (n-1)T(r, f) + o(T(r, f))$, a contradiction. □

Corollary 3.6.a: *let $a \in \mathbb{K}$ and $r > 0$. Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. $f \in \mathcal{M}_u(D)$, resp. $f \in \mathcal{M}^c(E)$). Then, f is transcendental over $\mathbb{K}(x)$.*

A function $h \in \mathcal{M}_b(D)$ is obviously small with respect to any function $f \in \mathcal{M}_u(D)$. So, we have the following corollary:

Corollary 3.6.b: *Let $a \in \mathbb{K}$ and $r > 0$ and let $f \in \mathcal{M}_u(D)$. Then, f is transcendental over $\mathcal{M}_b(D)$.*

Theorem 3.7 is known as Second Main Theorem on three small functions. It holds in the field $\mathbb{K}$ as well as in $\mathbb{C}$. But now, it also holds for meromorphic functions on E.

Theorem 3.7: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(D)$, resp. $f \in \mathcal{M}^c(E)$) and let $w_1, w_2, w_3 \in \mathcal{M}_f(\mathbb{K})$ (resp. $w_1, w_2, w_3 \in \mathcal{M}_f(D)$, resp. $w_1, w_2, w_3 \in \mathcal{M}_f(E)$) be pairwise distinct. Then $T(r, f) \leq \sum_{j=1}^{3} \overline{Z}(r, f - w_j) + o(T(r, f))$, (resp $T(r, f) \leq \sum_{j=1}^{3} \overline{Z}(r, f - w_j) + o(T(r, f))$, resp. $T_R(r, f) \leq \sum_{j=1}^{3} \overline{Z}_R(r, f - w_j) + o(T(r, f))$).*

Theorem 3.8: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(D)$, resp. $f \in \mathcal{M}^c(E)$ and let $w_1, w_2 \in \mathcal{M}_f(\mathbb{K})$ (resp. $w_1, w_2 \in \mathcal{M}_f(D)$, resp. $w_1, w_2 \in \mathcal{M}_f(E)$) be distinct. Then $T(r, f) \leq \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + \overline{N}(r, f) + o(T(r, f))$, (resp. $T(r, f) \leq \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + \overline{N}(r, f) + o(T(r, f))$, resp. $T_R(r, f) \leq \overline{Z}_R(r, f - w_1) + \overline{Z}_R(r, f - w_2) + \overline{N}_R(r, f) + o(T_R(r, f))$).*

Next, by setting $g = f - w_1$ and $w = w_1 + w_2$, we can write Corollary 3.8.a:

Corollary 3.8.a: *Let $g \in \mathcal{M}(\mathbb{K})$ (resp. $g \in \mathcal{M}_u(D)$, resp. $g \in \mathcal{M}^c(E)$) and let $w \in \mathcal{M}_g(\mathbb{K})$ (resp. $w \in \mathcal{M}_g(D)$, resp. $w \in \mathcal{M}_g(E)$). Then $T(r, g) \leq \overline{Z}(r, g) + \overline{Z}(r, g - w) + \overline{N}(r, g) + o(T(r, g))$, (resp. $T(r, g) \leq \overline{Z}(r, g) + \overline{Z}(r, g - w) + \overline{N}(r, g) + o(T(r, g))$, resp. $T_R(r, g) \leq \overline{Z}_R(r, g) + \overline{Z}_R(r, g - w) + \overline{N}(_R r, g) + o(T_R(r, g))$).*

Corollary 3.8.b: *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(D)$, (resp. $f \in \mathcal{A}^c(E)$) and let $w_1, w_2 \in \mathcal{A}_f(\mathbb{K})$ (resp. $w_1, w_2 \in \mathcal{A}_f(D)$, resp. $w_1, w_2 \in \mathcal{A}_f(E)$) be distinct. Then $T(r, f) \leq \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + o(T(r, f))$, (resp. $T(r, f) \leq \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + o(T(r, f))$), resp. $T_R(r, f) \leq \overline{Z}_R(r, f - w_1) + \overline{Z}_R(r, f - w_2) + o(T_R(r, f))$).*

And similarly to Corollary 3.8.a, we get Corollary 3.8.c :

Corollary 3.8.c: *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(D)$, resp. $f \in \mathcal{A}^c(E)$) and let $w \in \mathcal{A}_f(\mathbb{K})$ (resp. $w \in \mathcal{A}_f(D)$, resp. $w \in \mathcal{A}_f(E)$). Then $T(r, f) \leq \overline{Z}(r, f) +$*

$\overline{Z}(r, f-w)+o(T(r,f))$, *(resp.* $T(r,f) \leq \overline{Z}(r,f)+\overline{Z}(r,f-w)+o(T(r,f))$, *resp.* $T_R(r,f) \leq \overline{Z}_R(r,f)+\overline{Z}_R(r,f-w)+o(T_R(r,f))$*).*

4. New Properties of small functions

Here is now an application of that theory:

Theorem 4.1: *Let* $h,\ w \in \mathcal{A}_b(D)$ *and let* $m,\ n \in \mathbb{N}^*$ *be such that* $\min(m,n) \geq 2,\ \max(m,n) \geq 3$. *Then the functional equation*

$$(\mathcal{E}) \quad (g(x))^n = h(x)(f(x))^m + w(x)$$

has no solution (f,g) *in* $\mathcal{A}_u(D)$.

Proof. Without loss of generality, we can obviously assume $a=0$. Let $F(x) = g(x)^n$. Thanks to Corollary 3.8.c we can write

$$T(r,F) \leq \overline{Z}(r,F)+\overline{Z}(r,F-w)+o(T(r,F)).$$

Now, it appears that $\overline{Z}(r,F) \leq \frac{1}{n}Z(r,F)$. Moreover, since h is bounded, $Z(r,h)$ is bounded, hence $\overline{Z}(r,hf^m) \leq Z(r,f)+Z(r,h) = Z(r,f)+O(1)$, hence

$$\overline{Z}(r,hf^m) \leq \frac{1}{m}Z(r,hf^m)+O(1) = \frac{1}{m}Z(r,F)+O(1). \tag{1}$$

On the other hand, $Z(r,F) = Z(r,F-w)+O(1) = T(r,F)+O(1)$. Consequently, by (1), we can derive $T(r,F) \leq (\dfrac{1}{m}+\dfrac{1}{n})T(r,F)+o(T(r,F))$. Therefore we have $\dfrac{1}{m}+\dfrac{1}{n} \geq 1$, a contradiction to the hypothesis which implies $\dfrac{1}{m}+\dfrac{1}{n} \leq \dfrac{5}{6}$. □

Theorem 4.2: *Let* $f \in \mathcal{M}(\mathbb{K})$*be transcendental (resp.* $f \in \mathcal{M}_u(D)$*, resp.* $f \in \mathcal{M}^c(E)$*) and let* $w_j \in \mathcal{M}_f(\mathbb{K})$ $(j=1,...,q)$ *(resp.* $w_j \in \mathcal{M}_f(D)$*, resp.* $w_j \in \mathcal{M}_f(E)$ *) be* q *distinct small functions other than the constant* ∞*. Then*

$$qT(r,f) \leq 3\sum_{j=1}^{q}\overline{Z}(r,f-w_j)+o(T(r,f)),$$

(resp.

$$qT(r,f) \leq 3\sum_{j=1}^{q}\overline{Z}(r,f-w_j)+o(T(r,f)),$$

resp.

$$qT_R(r,f) \leq 3\sum_{j=1}^{q}\overline{Z}_R(r,f-w_j)+o(T_R(r,f))).$$

Moreover, if f *has finitely many poles in* $\mathbb{K}$ *(resp. in* D*, resp. in* E*), then*

$$qT(r,f) \leq 2\sum_{j=1}^{q}\overline{Z}(r,f-w_j)+o(T(r,f)),$$

(resp.

$$qT(r,f) \leq 2\sum_{j=1}^{q} \overline{Z}(r, f - w_j) + o(T(r,f)),$$

resp.

$$qT_R(r,f) \leq 2\sum_{j=1}^{q} \overline{Z}_R(r, f - w_j) + o(T_R(r,f))).$$

Definition: Let $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{M}_u(D)$), resp. $f,\ g \in \mathcal{M}^c(E)$). Then f and g will be said *to share a small function* $w \in \mathcal{M}(\mathbb{K})$ *(resp.* $w \in \mathcal{M}(D)$, *resp.* $w \in \mathcal{M}(C)$*), ignoring multiplicity* if $f(x) = w(x)$ implies $g(x) = w(x)$ and if $g(x) = w(x)$ implies $f(x) = w(x)$.

Theorem 4.3 : *Let* $f,\ g \in \mathcal{M}(\mathbb{K})$ *be transcendental (resp.* $f,\ g \in \mathcal{M}_u(D)$*, resp.* $f,\ g \in \mathcal{M}^c(E)$*) be distinct and share* q *distinct small functions ignoring multiplicity* $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ $(j = 1, ..., q)$ *(resp.* $w_j \in \mathcal{M}_f(D) \cap \mathcal{M}_g(D)$ $(j = 1, ..., q)$, *(resp.* $w_j \in \mathcal{M}_f(E) \cap \mathcal{M}_g(E)$ $(j = 1, ..., q)$*), other than the constant* ∞*. Then in* $\mathbb{K}$ *and in* D *we have*

$$\sum_{j=1}^{q} \overline{Z}(r, f - w_j) \leq \overline{Z}(r, f - g) + o(T(r,f)) + o(T(r,g))$$

and in E, we have

$$\sum_{j=1}^{q} \overline{Z}_R(r, f - w_j) \leq \overline{T}_R(r, f - g) + o(T_R(r,f)) + o(T_R(r,g)).$$

Proof. Suppose we are in $\mathbb{K}$ or in D. On one hand, when $f(x) = w_j(x)$, then $g(x) = w_j(x)$ hence $f(x) - g(x) = 0$. Consequently, we can check that

$$\sum_{j=1}^{q} \overline{Z}(r, f - w_j) \leq \overline{Z}(r, f - g) + \sum_{i \neq j} \overline{Z}(r, w_i - w_j)$$

But clearly, $\sum_{i \neq j} \overline{Z}(r, w_i - w_j) \leq o(T(r,f)) + o(T(r,g))$, which ends the proof. The proof is obviously similar if $f,\ g \in \mathcal{M}^c(E)$. □

The following Theorem 4.4 was proven for functions $f, g \in \mathcal{M}(\mathbb{K})$ and $f,\ g \in \mathcal{M}_u(D)$ in [6]. Here we can generalize the proof to $\mathcal{M}(E)$.

Theorem 4.4 : *Let* $f,\ g \in \mathcal{M}(\mathbb{K})$ *be transcendental (resp.* $f,\ g \in \mathcal{M}_u(D)$*, resp.* $f,\ g \in \mathcal{M}^c(E)$ *) be distinct and share 7 distinct small functions (other than the constant* ∞*) ignoring multiplicity,* $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ $(j = 1, ..., 7)$ *(resp.* $w_j \in \mathcal{M}_f(D) \cap \mathcal{M}_g(D)$*, resp.* $w_j \in \mathcal{M}_f(E) \cap \mathcal{M}_g(E)$ $(j = 1, ..., 7)$*,). Then* $f = g$*.*

Moreover, if f *and* g *have finitely many poles and share* 3 *distinct small functions (other than the constant* ∞*), ignoring multiplicity. then* $f = g$*.*

Proof. Suppose we are in $\mathbb{K}$ or in D. We put $M(r) = \max(T(r, f), T(r, g))$. Suppose that f and g are distinct and share q small function I.M. w_j, $(1 \leq j \leq q)$. By Theorem 3.10, we have

$$qT(r, f) \leq 3 \sum_{j=1}^{q} \overline{Z}(r, f - w_j) + o(T(r, f)).$$

But thanks to Theorem 4.3, we can derive

$$qT(r, f) \leq 3T(r, f - g) + o(T(r, f))$$

and similarly

$$qT(r, g) \leq 3T(r, f - g) + o(T(r, g))$$

hence

$$qM(r) \leq 3T(r, f - g) + o(M(r)). \tag{1}$$

By Theorem 2.2 and by Theorem 2.3, we can derive that

$$qM(r) \leq 3(T(r, f) + T(r, g)) + o(M(r)))$$

and hence $qM(r) \leq 6M(r) + o(M(r))$. Thus, this is impossible if $q \geq 7$ and hence the first statement of Theorem 4.4 is proved.

Suppose now that f and g have finitely many poles. By Theorems 2.2 and 2.3 and Relation (2) gives us

$$qM(r) \leq 2M(r) + o(M(r))$$

which is obviously absurd whenever $q \geq 3$ and proves that $f = g$ when f and g belong to $\mathcal{M}(\mathbb{K})$ as well as when f and g belong to $\mathcal{M}_u(d(0, R^-))$. The proof is similar if $f, g \in \mathcal{M}^c(E)$. □

Corollary 4.4.a : *Let* $f,\ g \in \mathcal{A}(\mathbb{K})$ *be transcendental (resp.* $f,\ g \in \mathcal{A}_u(D,$ *resp.* $f,\ g \in \mathcal{A}^c(E)$ *) be distinct and share 3 distinct small functions (other than the constant* ∞*), ignoring multiplicity,* $w_j \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ $(j = 1, 2, 3)$ *(resp.* $w_j \in \mathcal{A}_f(D) \cap \mathcal{A}_g(D)$, $(j = 1, 2, 3)$, *resp.* $w_j \in \mathcal{A}_f(E) \cap A_g(E)$ $(j = 1, 2, 3)$*). Then* $f = g$.

Remark: In complex analysis, thanks to Yamanoi's Theorem [12], it is easily seen that if two meromorphic functions in $\mathbb{C}$, f and g, share 5 small functions, then $f = g$. And if two entire functions in $\mathbb{C}$, f and g, share 4 small functions, then $f = g$. But apparently, the same process does not let us show that if two entire functions in $\mathbb{C}$, f and g, share 3 small functions, then $f = g$.

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